# 9.3 Taylor's Theorem

**Taylor's Theorem with Remainder**

If $f$ has derivatives of all orders in an open interval $I$ containing $a$, then for each positive integer $n$ and for each $x$ in $I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $c$ between $a$ and $x$. $R_n(x)$ is also called the **Lagrange form of the remainder**.

**Remainder Bounding Theorem**

If there are positive constants $M$ and $r$ such that $\left|f^{(n+1)}(t)\right| \leq Mr^{n+1}$ for all $t$ between $a$ and $x$, then the remainder $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{r^{n+1}|x-a|^{n+1}}{(n+1)!}.$$

If these conditions hold for every $n$ and all the other conditions of Taylor's Theorem are satisfied by $f$, then the series converges to $f(x)$.

**How to Use the Remainder Bounding Theorem**

1. **Identify** $f(x)$, the center $a$, and the value of $x$ you're approximating.
2. **Find a bound** on $|f^{(n+1)}(t)|$ for all $t$ between $a$ and $x$, i.e., pick constants $M$ and $r$ so that $|f^{(n+1)}(t)| \leq Mr^{n+1}$.
3. **Plug into the bound:**

$$|R_n(x)| \leq M \cdot \frac{r^{n+1}|x-a|^{n+1}}{(n+1)!}.$$

4. **Solve for** $n$ (if you need a target accuracy): set the right side $<$ your error tolerance and find the smallest $n$ that works.
5. **Convergence is automatic** once you've found $M$ and $r$ that work for all $n$, the $(n+1)!$ in the denominator dominates, so $R_n(x) \to 0$ and the Taylor series converges to $f(x)$.

**Alternating Series Error Bound**

Suppose the terms of a series have the following three properties:

1. the terms alternate in sign;
2. the terms decrease in absolute value;
3. the terms approach 0 as a limit.

Then the truncation error after $n$ terms is less than the absolute value of the $(n+1)$st term.

1) Find a formula for the truncation error if we use $P_6(x)$ to approximate $\frac{1}{1-2x}$ on the interval $(-\frac{1}{2}, \frac{1}{2})$.

## Solution

The center of the interval is 0.

Hence we use the Maclaurin series

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \dots$$

(i.e. infinite geometric series with $a_1 = 1$ and $r = 2x$, $-1 < 2x < 1 \rightarrow -\frac{1}{2} < x < \frac{1}{2}$)

and hence

$$P_6(x) = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + 64x^6$$

$$\text{The truncation error} = \frac{1}{1-2x} - P_6(x)$$

$$= (2x)^7 + (2x)^8 + (2x)^9 + \dots$$

$$= \boxed{\frac{128x^7}{1-2x}}$$

(i.e. $\frac{a_1}{1-r}$ where $a_1 = (2x)^7$, $r = 2x$, $-1 < 2x < 1$)

| $x$ | $y = \frac{1}{1-2x}$ | $y \approx P_6(x)$ | error $= \frac{1}{1-2x} - P_6(x)$ | error $= \frac{128x^7}{1-2x}$ |
|---|---|---|---|---|
| 0 | 1 | 1.000000 | 0.000000 | 0.000000 |
| 0.1 | 1.25 | 1.249984 | 0.000016 | 0.000016 |
| -0.3 | 0.625 | 0.642496 | -0.017496 | -0.017496 |
| -0.49 | 0.505050505 | 0.943498 | -0.438447 | -0.438447 |
| 0.49 | 50 | 6.593723 | 43.406277 | 43.406277 |

2) Use the Remainder Bounding Theorem to prove that the Maclaurin series for $f(x) = \sin 5x$ converges to $f(x)$ for all $\mathbb{R}$.

Solution

By Taylor's Theorem,

$R_n = f^{(n+1)}(c) \dfrac{x^{n+1}}{(n+1)!}$ is the remainder term (i.e. the truncation error when $P_n(x)$ is used to approximate $f(x)$).

$|f^{(n+1)}(x)|$ is either $|5^{n+1} \sin x|$ or $|5^{n+1} \cos x|$

$f'(x) = 5\cos 5x$
$f''(x) = -5^2 \sin 5x$
$f'''(x) = -5^3 \cos 5x$
$f^{(4)}(x) = 5^4 \sin 5x$
$\vdots$

Whether $t \in [0,x]$ or $[x,0]$,

$-1 \cdot 5^{n+1} \leq f^{(n+1)}(t) \leq 1 \cdot 5^{n+1}$

Thus, $|f^{(n+1)}(t)| \leq 5^{n+1}$ (That is $M=1$, $r=5$)

Maclaurin series means $a=0$

Thus by the Remainder Bounding Theorem,

$|R_n(x)| \leq 5^{n+1} \cdot \dfrac{|x|^{n+1}}{(n+1)!}$ (which goes to zero as $n \to \infty$ since $(n+1)!$ grows much faster than $5^{n+1}|x|^{n+1}$)

and thus the Maclaurin series converges to $f(x)$ for all $\mathbb{R}$.

3) Suppose you use $P_2(x) = 1 - \frac{x^2}{2}$ to approximate $\cos(0.5)$.
Give a bound on the error based on...

a) the alternating series bound

<u>Solution</u>

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ when $n=1$, we get $P_2(x)$

Notice that the terms of this Maclaurin Series:

✓alternate in signs

✓decrease in absolute value

✓approach zero as $n \to \infty$

Therefore, by the Alternating Series Error Bound,
if we use $P_2(x) \approx \cos x$, then the error $< \left| \frac{x^4}{4!} \right|$ when $n=2$, we get the term in the 4th degree

Hence $|P_2(0.5) - \cos(0.5)| < \frac{0.5^4}{4!} \approx 0.0026042$

b) the Lagrange Error Formula

<u>Solution</u>

By the Lagrange Error Formula, the truncation error from using $P_2(x)$ is $\left| f'''(c) \cdot \frac{x^3}{3!} \right|$ i.e. $\left| P_2(x) - \cos x \right| = \left| f'''(c) \cdot \frac{x^3}{3!} \right|$

$f'''(t) = \sin t$

Whether $t \in [0, x]$ or $[x, 0]$,

$|f'''(t)| \leq 1$. That is $M = 1$.

Thus, by the Remainder Bounding theorem,

$\left| f'''(c) \cdot \frac{x^3}{3!} \right| \leq 1 \cdot \frac{|x|^3}{3!}$

That is, the error $|P_2(0.5) - \cos 0.5| \leq \frac{|0.5|^3}{3!} = 0.0208\bar{3}$

4) The hyperbolic sine and hyperbolic cosine functions are defined as:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Find the Maclaurin series generated by sinhx and coshx.

<u>Solution</u>

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \quad + \frac{(-x)^n}{n!} + \cdots$$

$$-e^{-x} = -1 + x - \frac{x^2}{2!} + \cdots + \frac{(-1)^{n+1} x^n}{n!} + \cdots$$

$$e^x - e^{-x} = 2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\therefore \boxed{\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}} \quad , \quad \boxed{\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

5) Use the Remainder Bounding Theorem to prove that $\cosh x$ equals its Maclaurin series for all real numbers $x$.

<u>Proof</u>

We first need to find $M$ such that

$$\left|f^{(n+1)}(t)\right| \leq M \quad \text{for all } t \in [x,0] \text{ or } [0,x]$$

$f^{(n+1)}(x)$ is either $\cosh x$ or $\sinh x$

$f(x) = \frac{1}{2}(e^x + e^{-x})$
$f'(x) = \frac{1}{2}(e^x - e^{-x})$
$f''(x) = \frac{1}{2}(e^x + e^{-x})$
$f'''(x) = \frac{1}{2}(e^x - e^{-x})$
$\vdots$

If $t \in [0,x]$,

$$\cosh t = \frac{1}{2}(e^t + e^{-t}) \leq \frac{1}{2}(e^t + e^t) = e^t \leq e^x$$

$$\sinh t = \frac{1}{2}(e^t - e^{-t}) \leq \frac{1}{2}(e^t + e^{-t}) = \cosh t \leq e^x$$

Thus

$$\left|f^{(n+1)}(t)\right| \leq e^x \quad \text{(that is, } M = e^x\text{)}$$

If $t \in [x,0]$,

$$\cosh t = \frac{1}{2}(e^t + e^{-t}) \leq \frac{1}{2}(e^{-t} + e^{-t}) = e^{-t} \leq e^{-x}$$

$$\sinh t = \frac{1}{2}(e^t - e^{-t}) \leq \frac{1}{2}(e^t + e^{-t}) = \cosh t \leq e^{-x}$$

Thus

$$\left|f^{(n+1)}(t)\right| \leq e^{-x} \quad \text{(that is, } M = e^{-x}\text{)}$$

Thus, in both cases, the Maclaurin series converges to $f(x) = \cosh x$ since, by the Remainder Bounding Theorem,

$$|R_n(x)| < M \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

6) Find the linearization and the quadratic approximation of $f(x) = \sec x$ at $x = 0$.

Solution

The linearization of $f(x)$ at $x=0$

$$\begin{aligned} L(x) &= f(0) + f'(0)x \\ &= 1 + \overbrace{\sec 0}^{1}\,\overbrace{\tan 0}^{0}\, x \\ &= 1 \end{aligned}$$

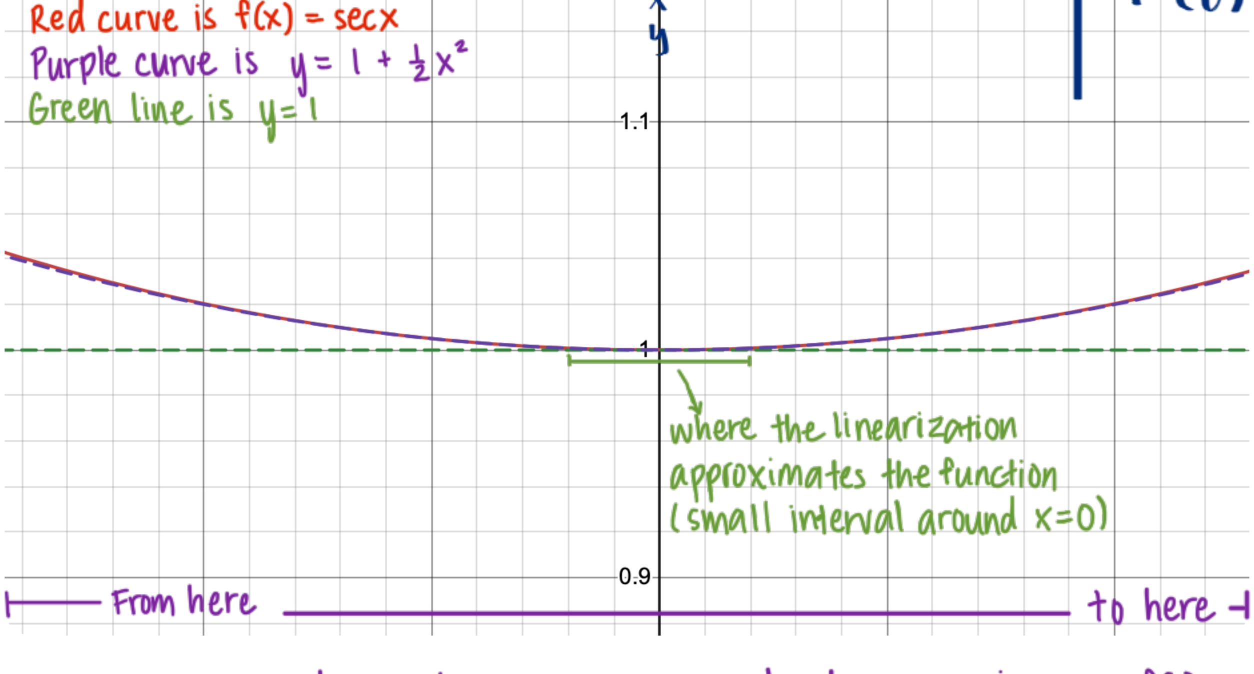
The quadratic approximation of $f(x)$ at $x=0$

$$\begin{aligned} P_2(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 \\ &= 1 + 0x + \frac{1}{2}x^2 \\ &= 1 + \frac{1}{2}x^2 \end{aligned}$$

$f'(x) = \sec x \tan x$
$f'(0) = 0$

$f''(x) = \sec x \cdot \sec^2 x + \tan x \sec x \tan x$

$f''(0) = 1$




is where the purple curve closely approximates f(x).

7) Given $f(x) = \sin(5x + \frac{\pi}{4})$, let $P_3(x)$ be the third degree Taylor polynomial for f about $x=0$.

a) Find $P(x)$.

Solution

$$P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$f(0) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$f'(x) = 5\cos(5x + \frac{\pi}{4})$
$f'(0) = 5\cos(\frac{\pi}{4}) = 5 \cdot \frac{\sqrt{2}}{2}$

$f''(x) = -5^2 \sin(5x + \frac{\pi}{4})$
$f''(0) = -5^2 \cdot \frac{\sqrt{2}}{2}$

$f'''(x) = -5^3 \cos(5x + \frac{\pi}{4})$
$f'''(0) = -5^3 \cdot \frac{\sqrt{2}}{2}$

$$\therefore \boxed{P_3(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{5^2\sqrt{2}}{4}x^2 - \frac{5^3\sqrt{2}}{12}x^3}$$

The coefficient of the term in $x^n$ is $\boxed{5^n \cdot \frac{\sin(\frac{\pi}{4} + n\cdot\frac{\pi}{2})}{n!}}$

b) Use the Lagrange error bound to show that

$$\left| f\left(\tfrac{1}{10}\right) - P_3\left(\tfrac{1}{10}\right) \right| < \frac{1}{100}$$

Solution

The Lagrange error when using $P_3(x)$ to approximate $f(x)$ is

$$R_n(x) = \left| f^{(4)}(c) \cdot \frac{x^4}{4!} \right| \qquad f^{(4)}(t) = 5^4 \sin(5t + \tfrac{\pi}{4})$$

Whether $t \in [0,x]$ or $[x,0]$,

$f^{(4)}(t) \leq 5^4$ since $\sin(5t + \frac{\pi}{4}) \leq 1$

That is $M = 5^4$

By the Remainder Bounding theorem,

$$|R_n(x)| \leq M \cdot \frac{|x|^4}{4!} = \frac{5^4 |x|^4}{4!}$$

When $x = \frac{1}{10}$, $\left|R_n\left(\frac{1}{10}\right)\right| = \left|f\left(\frac{1}{10}\right) - P_3\left(\frac{1}{10}\right)\right| \leq 5^4 \cdot \frac{1}{10^4} \cdot \frac{1}{4!}$

$$= \frac{1}{2^4} \cdot \frac{1}{4!}$$

$$= \frac{1}{384}$$

$$< \frac{1}{100}$$

c) Let $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for $G$ about $x=0$.

<u>Solution</u>

$\frac{d}{dx} G(x) = f(x)$ by the Fundamental Theorem of Calculus

$\approx \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} x - \frac{5^2\sqrt{2}}{4} x^2$ when $x$ is near $0$

Thus $G(x) \approx \int_0^x \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} t - \frac{5^2\sqrt{2}}{4} t^2 \, dt$

$$= \frac{\sqrt{2}}{2} t + \frac{5\sqrt{2}}{4} t^2 - \frac{5^2\sqrt{2}}{12} t^3 \Bigg|_{t=0}^{t=x}$$

$$= \boxed{\frac{\sqrt{2}}{2} x + \frac{5\sqrt{2}}{4} x^2 - \frac{5^2\sqrt{2}}{12} x^3}$$

**Resource**

Finney, Ross L., Franklin D. Demana, Bert K. Waits, Daniel Kennedy, and David M. Bressoud. *Calculus: Graphical, Numerical, Algebraic*. 5th ed., AP ed., Pearson, 2016.