9.2 Taylor Series

- The Taylor Polynomial of order 4 for f at x=0:

$$P_4(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

- Maclaurin series for
  - $\frac{1}{1-x}$
  - $\sin x$
  - $\tan^{-1}(x)$
  - $\frac{1}{1+x}$
  - $\cos x$
  - e^x
  - $\ln(1+x)$

- Taylor series of f at x=a: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

1) Construct a 4th order Taylor polynomial at $x=0$ for

$$f(x) = \sqrt{1+x^2}$$

Solution

Let $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$
be the 4th order Taylor polynomial of $f(x)$ at $x=0$

To find a_0, a_1, a_2, a_3, a_4, we use the ff eq:

$$P(0) = f(0)$$
$$P'(0) = f'(0)$$
$$P''(0) = f''(0)$$
$$P'''(0) = f'''(0)$$
$$P^{(4)}(0) = f^{(4)}(0)$$

Derivatives of $P(x)$

$$P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$
$$P''(x) = 2a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 a_4 x^2$$
$$P'''(x) = 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4 x$$
$$P^{(4)}(x) = 4 \cdot 3 \cdot 2 \cdot a_4$$

$$P'(0) = 1!\, a_1$$
$$P''(0) = 2!\, a_2$$
$$P'''(0) = 3!\, a_3$$
$$P^{(4)}(0) = 4!\, a_4$$

$(P^{(n)}(0) = n!\, a_n)$

Derivatives of $f(x)$

$$f'(x) = \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x$$
$$= x(1+x^2)^{-\frac{1}{2}} \longrightarrow f'(0) = 0$$

$$f''(x) = x\left(-\frac{1}{2}\right)(1+x^2)^{-\frac{3}{2}}(2x) + (1+x^2)^{-\frac{1}{2}}$$
$$= -x^2(1+x^2)^{-\frac{3}{2}} + (1+x^2)^{-\frac{1}{2}} \longrightarrow f''(0) = 1$$

$$f'''(x) = -x^2\left(-\frac{3}{2}\right)(1+x^2)^{-\frac{5}{2}}(2x) + (1+x^2)^{-\frac{3}{2}}(-2x) + \left(-\frac{1}{2}\right)(1+x^2)^{-\frac{3}{2}}(2x)$$

$$= 3x^3(1+x^2)^{-\frac{5}{2}} - 2x(1+x^2)^{-\frac{3}{2}} - x(1+x^2)^{-\frac{3}{2}}$$

$$= 3x^3(1+x^2)^{-\frac{5}{2}} - 3x(1+x^2)^{-\frac{3}{2}} \longrightarrow f'''(0) = 0$$

$$f^{(4)}(x) = 3x^3\left(-\frac{5}{2}\right)(1+x^2)^{-\frac{7}{2}}(2x) + (1+x^2)^{-\frac{5}{2}} \cdot 9x^2 + (-3x)\left(-\frac{3}{2}\right)(1+x^2)^{-\frac{5}{2}}(2x) + (1+x^2)^{-\frac{3}{2}}(-3)$$

$$= -15x^4(1+x^2)^{-\frac{7}{2}} + 18x^2(1+x^2)^{-\frac{5}{2}} - 3(1+x^2)^{-\frac{3}{2}} \rightarrow f^{(4)}(0) = -3$$

So,

$$\boxed{\begin{aligned} a_0 &= f(0) \\ 1!\,a_1 &= f'(0) \\ 2!\,a_2 &= f''(0) \\ 3!\,a_3 &= f'''(0) \\ 4!\,a_4 &= f^{(4)}(0) \end{aligned}}
\begin{aligned} &\rightarrow a_0 = 1 \\ &\rightarrow a_1 = 0 \\ &\rightarrow 2a_2 = 1 \rightarrow a_2 = \tfrac{1}{2} \\ &\rightarrow 6a_3 = 0 \rightarrow a_3 = 0 \\ &\rightarrow 24a_4 = -3 \rightarrow a_4 = -\tfrac{1}{8} \end{aligned}$$

$a_0 = f(0)$

$a_1 = \frac{f'(0)}{1!}$

$a_2 = \frac{f''(0)}{2!}$

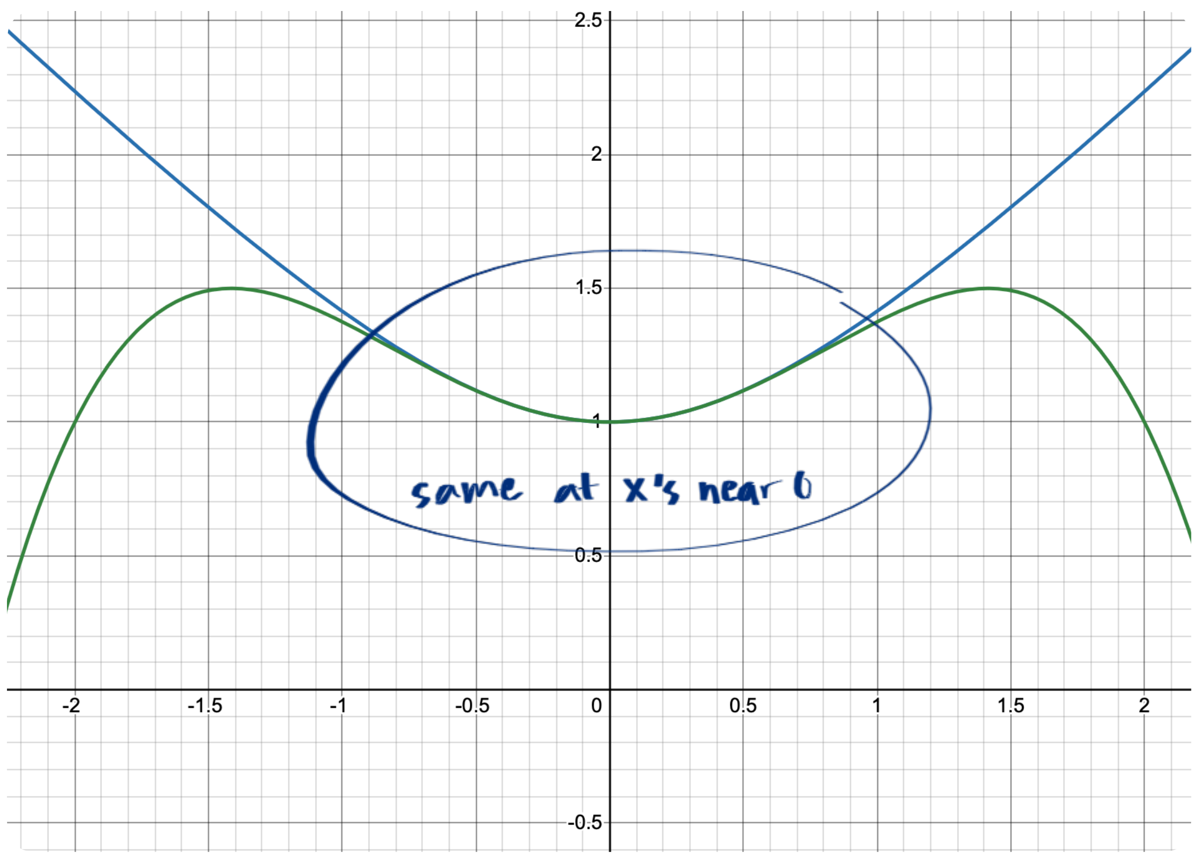
$a_3 = \frac{f'''(0)}{3!}$

$a_4 = \frac{f^{(4)}(0)}{4!}$

Answer: $\boxed{P(x) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4}$

$f(x)$: $\sqrt{1+x^2}$

$P_4(x)$: $1 + \frac{1}{2}x^2 - \frac{1}{8}x^4$



2) Construct the fifth-order Taylor polynomial and the Taylor series at $x=0$ for

$f(x) = e^{1-x}$

<u>Solution</u>

$$P_5(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

is the Taylor Poly. of order 5 for $f(x)$.

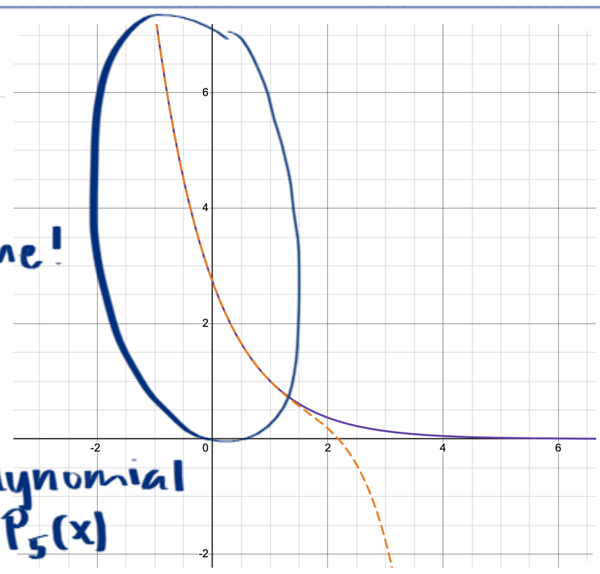
$f(x) = e^{1-x}$	$f(0) = e$
$f'(x) = -e^{1-x}$	$f'(0) = -e$
$f''(x) = e^{1-x}$	$f''(0) = e$
$f'''(x) = -e^{1-x}$	$f'''(0) = -e$
$f^{(4)}(x) = e^{1-x}$	$f^{(4)}(0) = e$
$f^{(5)}(x) = -e^{1-x}$	$f^{(5)}(0) = -e$

Ans:

$$\boxed{P_5(x) = e - \frac{e}{1!}x + \frac{e}{2!}x^2 - \frac{e}{3!}x^3 + \frac{e}{4!}x^4 - \frac{e}{5!}x^5}$$

$f(x)$: $e^{(1-x)}$

$P_5(x)$: $e - ex + \frac{e}{2!}x^2 - \frac{e}{3!}x^3 + \frac{e}{4!}x^4 - \frac{e}{5!}x^5$



same!

Isn't this cool?

non polynomial e^{1-x} $\longrightarrow$ polynomial $P_5(x)$

Therefore,

$$e^{1-x} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k e}{k!} x^k}$$

The Taylor series for e^{1-x} at $x=0$

a.k.a the Maclaurin Series generated by e^{1-x}

3) Given $f(x) = \ln(1-x)$, use the table of Maclaurin series

a) to construct the first 3 non-zero terms

b) to construct the general term

c) to construct the interval of convergence

Solution

From the table, $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, $-1 < x \le 1$

seems most useful.

$$\ln(1-x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-x)^n}{n} , \quad -1 < -x \le 1$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} (-1)^n \frac{x^n}{n} , \quad 1 > x \ge -1$$

$$= \sum_{n=1}^{\infty} -\frac{x^n}{n} , \quad -1 \le x < 1$$

Answers:

a) $-x - \frac{x^2}{2} - \frac{x^3}{3}$

b) $\sum_{n=1}^{\infty} -\frac{x^n}{n}$

c) $x \in [-1, 1)$

4) Given $f(x) = x^2\cos x$, use the table of Maclaurin series

a) to construct the first 3 non-zero terms

b) to construct the general term

c) to construct the interval of convergence

Solution

From the table, $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, $x \in \mathbb{R}$

is most useful:

$$x^2\cos x = \sum_{n=0}^{\infty} x^2 (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!} \quad , \quad x \in \mathbb{R}$$

Answers:

a) $x^2 - \frac{x^4}{2!} + \frac{x^6}{4!}$

b) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!}$

c) $x \in \mathbb{R}$

5) Find the Taylor series generated by

$$f(x) = \frac{1}{x+1} \quad \text{at} \quad x=2.$$

Solution

$$\frac{1}{x+1} = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \dots$$

The derivatives of $f(x)$:

$$f'(x) = -(x+1)^{-2} \qquad f'(2) = -1!\,3^{-2}$$

$$f''(x) = 2(x+1)^{-3} \qquad f''(2) = +2!\,3^{-3}$$

$$f'''(x) = -3\cdot 2(x+1)^{-4} \qquad f'''(2) = -3!\,3^{-4}$$

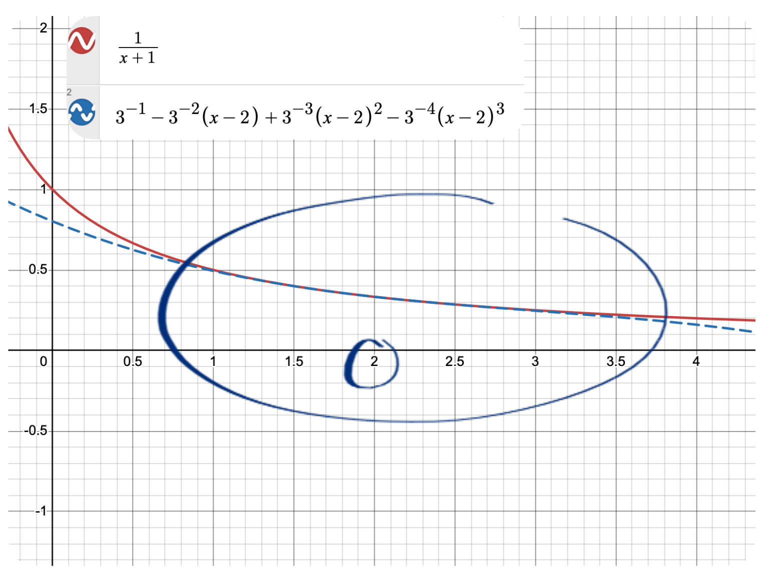
$$f^{(4)}(x) = 4\cdot 3\cdot 2(x+1)^{-5} \qquad f^{(4)}(2) = +4!\,3^{-5}$$

$$\vdots \qquad \vdots$$

Therefore

$$\frac{1}{x+1} = 3^{-1} - 3^{-2}(x-2) + 3^{-3}(x-2)^2 - 3^{-4}(x-2)^3 + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \cdot 3^{n-1} (x-2)^n}$$



b) Find the Taylor polynomial of order 3 generated by $f(x) = \cos x$ at $x = \frac{\pi}{4}$

Solution

$$P_3(x) = f\left(\tfrac{\pi}{4}\right) + \frac{f'\left(\frac{\pi}{4}\right)}{1!}\left(x-\tfrac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\tfrac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\tfrac{\pi}{4}\right)^3$$

$f(x) = \cos x \longrightarrow f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right)$

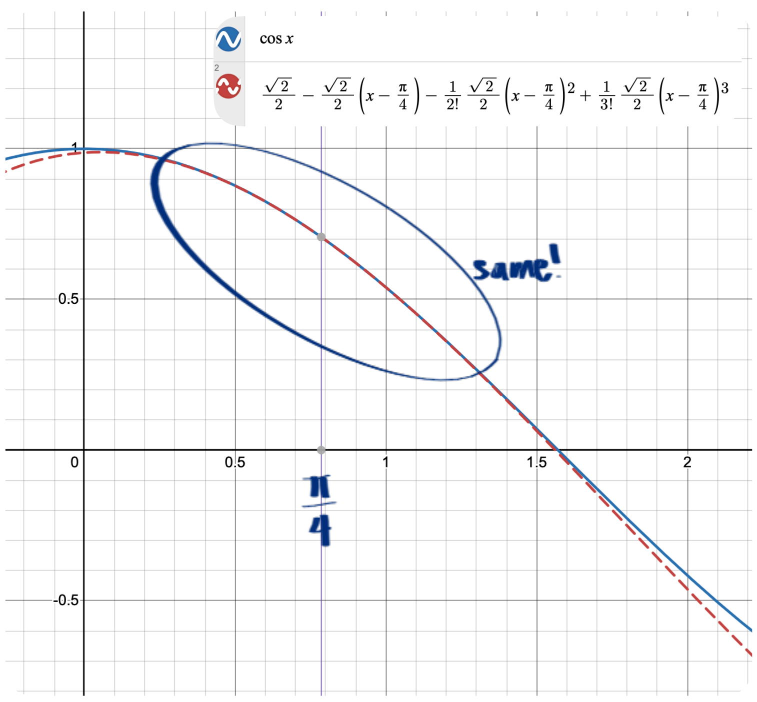
$f'(x) = -\sin x \qquad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} = \cos\left(\frac{3\pi}{4}\right)$

$f''(x) = -\cos x \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} = \cos\left(\frac{5\pi}{4}\right)$

$f'''(x) = \sin x \qquad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\left(\frac{7\pi}{4}\right)$

$$P_3(x) = \frac{\sqrt{2}}{2} - \frac{1}{1!}\frac{\sqrt{2}}{2}\left(x-\tfrac{\pi}{4}\right) - \frac{1}{2!}\frac{\sqrt{2}}{2}\left(x-\tfrac{\pi}{4}\right)^2 + \frac{1}{3!}\frac{\sqrt{2}}{2}\left(x-\tfrac{\pi}{4}\right)^3$$

$$= \boxed{\sum_{n=0}^{3} \frac{1}{n!} \cdot \cos\left(\frac{(2n+1)\pi}{4}\right)\left(x-\tfrac{\pi}{4}\right)^n}$$



7) Let $f(t) = \frac{2}{1-t^2}$ and $G(x) = \int_0^x f(t)\,dt$

a) Find the first four terms and the general term for the Maclaurin series generated by f.

Solution

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots, \quad -1 < t < 1$$

So,

$$\frac{2}{1-t^2} = 2 \cdot \frac{1}{1-t^2}$$

$$= 2(1 + t^2 + t^4 + t^6 + \dots)$$

$$= \boxed{2 + 2t^2 + 2t^4 + 2t^6 + \dots} = \boxed{\sum_{n=0}^{\infty} 2t^{2n}}, \quad -1 < t < 1$$

b) Find the first four nonzero terms and the Maclaurin series for G.

Solution

$$G(x) = \int_0^x f(t)\,dt$$

$$= \int_0^x 2 + 2t^2 + 2t^4 + 2t^6 + \dots\,dt$$

$$= 2t + 2\frac{t^3}{3} + 2\frac{t^5}{5} + 2\frac{t^7}{7} + \dots \Big|_{t=0}^{t=x}$$

$$= \boxed{2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + 2\frac{x^7}{7} + \dots}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}, \quad -1 < x < 1}$$

If you're comfortable with the theorem in Ch. 9.1:

$$\int_{t=0}^{t=x} \sum_{n=0}^{\infty} C_n t^n\,dt = \sum_{n=0}^{\infty} C_n \frac{x^{n+1}}{n+1}$$ (when x is inside the interval of convergence)

then the result can be immediate.

8) Show that if $f''(x)$ exists over the domain of f and the graph of $f(x)$ has an inflection point at $x=a$, then the linearization of f at $x=a$ is also the second-order Taylor polynomial of f at $x=a$.

Solution

Linearization of f at $x=a$:

$$L(x) = f(a) + f'(a)(x-a)$$

· Ch 4.5

Second-order Taylor polynomial of f at $x=a$:

$$P_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

However, $f''(a)=0$ since $f''(x)$ exists and $(a, f(a))$ is an inflection point.

Therefore,

$$\boxed{L(x) = P_2(x) = f(a) + f'(a)(x-a)} \quad //$$

(The implication of this is that, near inflection points, the tangent line practically coincides/overlaps the graph of $f(x)$).

(the tangent line provides an especially accurate local approximation near an inflection point)

9) To what number does

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

converge when $x=1$? when $x=-1$?

<u>Solution</u>

(through memorization of the Maclaurin series table),
we recognize that:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \tan^{-1}(x)$$

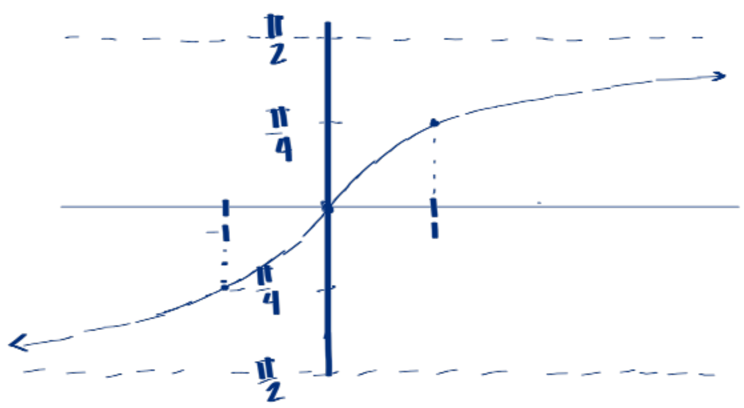
with interval of convergence $-1 \leq x \leq 1$.

Therefore,

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1} = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1} = \tan^{-1}(-1) = -\frac{\pi}{4}$$

By now, please know the graph of arctan(x)



Definition: Taylor Series Generated by f at $x = 0$ (Maclaurin Series)

Let f be a function with derivatives of all orders throughout some open interval containing 0. Then the Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.$$

This series is also called the **Maclaurin series** generated by f.

The Partial Sum

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

is the **Taylor polynomial of order** n for f at $x = 0$.

Definition: Taylor Series Generated by f at $x = a$

Let f be a function with derivatives of all orders throughout some open interval containing a. Then the Taylor series generated by f at $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The Partial Sum

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the **Taylor polynomial of order** n for f at $x = a$.

Maclaurin Series (Taylor Series at $x = 0$)

Function	Series Expansion	Sigma Notation	Interval of Convergence
$\frac{1}{1-x}$	$1 + x + x^2 + \cdots + x^n + \cdots$	$\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
$\frac{1}{1+x}$	$1 - x + x^2 - \cdots + (-x)^n + \cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$(-1, 1)$
e^x	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$(-1, 1]$
$\tan^{-1} x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$[-1, 1]$

Resources

1. **Finney, Ross L., Franklin D. Demana, Bert K. Waits, Daniel Kennedy, & David M. Bressoud**
   Calculus: Graphical, Numerical, Algebraic (AP® Edition, 5th Edition)
   Pearson Education, 2016.
2. **Stewart, James; Daniel Clegg; & Saleem Watson**
   Calculus: Early Transcendentals (9th Edition)
   Cengage Learning, 2021.
   ISBN: 978-1-337-61392-7
3. **Khan Academy**
   AP® Calculus BC
   https://www.khanacademy.org/math/ap-calculus-bc ↗