9.1 Power Series

- "Basic" methods to determine whether an infinite series converges or diverges.
  - ☐ via limit of the seq. of partial sums
  - ☐ via Knowledge of infinite geometric series
  - ☐ via nth term test for divergence

- Basic methods to find a function representation for a power series (and vice versa: a power series rep for a function)
  - ☐ emphasis on stating the domain (interval of convergence)
  - ☐ via infinite geometric series formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \; |x|<1$$

  - ☐ via differentiation

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=1}^{\infty} c_n \cdot n \cdot x^{n-1} \quad , \; x \in \text{interval of conv.}$$

  - ☐ via integration

$$\int_{a}^{x} \sum_{n=0}^{\infty} c_n (t-a)^n \, dt = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

1) Write an expression for a_n

a) $\sum_{n=0}^{\infty} a_n = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$

<u>Answer</u>: $a_n = \boxed{\frac{1}{3^n}}$

$$\underset{n=0}{1} + \underset{n=1}{\frac{1}{3}} + \underset{n=2}{\frac{1}{9}} + \underset{n=3}{\frac{1}{27}} + \underset{n=4}{\frac{1}{81}} + \dots$$

b) $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

<u>Answer</u>: $\boxed{a_n = (-1)^{n-1} \frac{1}{n}}$

★ $(-1)^{\text{even}} = +1$

$(-1)^{\text{odd}} = -1$

2) Determine whether the series converges by finding the limit of partial sums.

a) $1 + 1.1 + 1.11 + 1.111 + \dots$

<u>Solution</u>

$S_1 = \underline{1.1}$

$S_2 = 1.1 + 1.11 = \underline{2.21}$

$S_3 = \underbrace{1.1 + 1.11}_{S_2} + 1.111 = \underline{3.321}$

$S_4 = \underbrace{S_3}_{} + 1.1111 = \underline{4.4321}$

Sn is unbounded

$S_n \to \infty$ as $n \to \infty$

∴ The series diverges

b) $3 + 0.5 + 0.05 + 0.005 + \dots$

<u>Solution</u>

$S_1 = 3$

$S_2 = 3.5$

$S_3 = 3.55$

$S_4 = 3.555$

$S_n \to 3\frac{5}{9}$ as $n \to \infty$

∴ The series converges

3) Let $S_n = \dfrac{n+1}{2n+5}$ be the nth partial sum of the series $\sum_{n=0}^{\infty} a_n$.
Write a rule for a_n.

Solution

Write the solution with me by filling the blanks

$$a_n = S_n - \boxed{\phantom{S_{n-1}}}$$

$$= \frac{n+1}{2n+5} - \frac{\boxed{}}{\boxed{}}$$

$$= \boxed{\frac{3}{4n^2+16n+15}}$$

4) Tell whether the infinite series converges or diverges.

a) $1 - 2 + 3 - 4 + 5 - \cdots + (-1)^n(n+1) + \cdots$

Solution

Find limit of partial sums

$S_1 = 1$
$S_2 = 1 - 2 = -1$
$S_3 = -1 + 3 = 2$
$S_4 = 2 - 4 = -2$
$S_5 = -2 + 5 = 3$
$S_6 = 3 - 6 = -3$

$\{1, -1, 2, -2, 3, -3, \ldots\}$

$\therefore$ Diverges since the sequence of partial sums diverges. (specifically, the seq of partial sums is oscillating and also unbounded).

Answer to #3

$$a_n = S_n - \boxed{S_{n-1}}$$

$$= \frac{n+1}{2n+5} - \frac{\boxed{(n-1)+1}}{\boxed{2(n-1)+5}}$$

$$= \boxed{\frac{3}{4n^2+16n+15}}$$

b) $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)\left(\frac{5}{4}\right)^n$

Answer: Diverges since this is an infinite geometric series with $r = \frac{5}{4} > 1$ (The infinite series is unbounded).

c) $3 - 0.3 + 0.03 - 0.003 + 0.0003 - \ldots + 3(-0.1)^n + \ldots$

Answer: Converges since this is an infinite geometric series with $r = -0.1$, $-1 < -0.1 < 1$

d) $\sum_{n=0}^{\infty} \sin^n\left(\frac{\pi}{4} + n\pi\right)$

Solution

$$\underset{n=0}{1} + \underset{n=1}{\sin\left(1\tfrac{1}{4}\pi\right)} + \underset{n=2}{\sin^2\left(2\tfrac{1}{4}\pi\right)} + \underset{n=3}{\sin^3\left(3\tfrac{1}{4}\pi\right)} + \ldots$$

$$= 1 + -\frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} \left(-\frac{\sqrt{2}}{2}\right)^n$$

$\therefore$ Converges since it is an infinite geometric series with $r = -\frac{\sqrt{2}}{2}$, $-1 < -\frac{\sqrt{2}}{2} < 1$

5) Find the interval of convergence and the function of x represented by the geometric series

a) $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

b) $\sum_{n=0}^{\infty} \tan^n x$

Solution (a)

$3 + 3\left(\frac{x-1}{2}\right) + 3\left(\frac{x-1}{2}\right)^2 + \ldots$

$r = \frac{x-1}{2}$, $a_1 = 3$

$\therefore$ Interval of convergence:

$-1 < \frac{x-1}{2} < 1$

$-2 < x-1 < 2$

$-1 < x < 3$

$\therefore \sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

$= \frac{3}{1 - \frac{x-1}{2}}$

Recall: $S_\infty = \frac{a_1}{1-r}$, $|r|<1$

$= \frac{3}{\frac{2-x+1}{2}}$

$= \boxed{\frac{6}{3-x}, \quad -1<x<3}$

Solution (b)

$1 + \tan x + \tan^2 x + \ldots$

$r = \tan x$

$\therefore$ Interval of convergence:

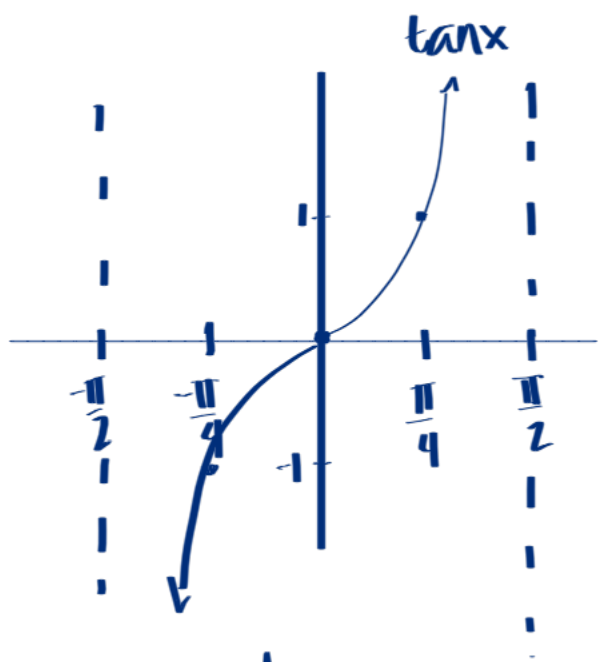
$-1 < \tan x < 1$

$-\frac{\pi}{4} < x < \frac{\pi}{4}$

$-\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi$, $k \in \mathbb{Z}$

$\therefore \sum_{n=0}^{\infty} \tan^n x = \frac{1}{1-\tan x}$,

$-\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi$
$k \in \mathbb{Z}$



period : π

6) Use differentiation to find a power series for

$$f(x) = \frac{2}{(1-x)^3}$$

Solution

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad |x|<1$$

always state the domain (interval of convergence)

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n$$

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \sum_{n=1}^{\infty} (n+1)(n) x^{n-1}$$

$$\frac{2}{(1-x)^3} = \boxed{\sum_{n=0}^{\infty} (n+2)(n+1) x^n \quad , \quad |x|<1}$$

$$\left\{ \begin{array}{l} \int 2(1-x)^{-3}\,dx = -2\,\frac{(1-x)^{-2}}{-2} = (1-x)^{-2} \\ \int (1-x)^{-2}\,dx = -\frac{(1-x)^{-1}}{-1} = \frac{1}{1-x} \\ \text{This means that } \boxed{\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right)} \end{array} \right.$$

Theorem:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) \boxed{= \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}}$$

Alternatively: $\sum_{n=2}^{\infty} n(n-1) x^{n-2}$

7) a) Find a power series that represents $\frac{1}{x}$ on $(0, 2)$.

Solution

$$\frac{1}{x} = \frac{1}{1+(x-1)}$$

we need to force out the expression $(x-1)$ because the center of $(0,2)$ is $a=1$.

$$= \frac{1}{1-\underbrace{(-(x-1))}_{r}}$$

write in form: $\frac{1}{1-r}$, $|r|<1$

$$= \sum_{n=0}^{\infty} (-(x-1))^n \quad ,$$

$$-1 < \overbrace{-(x-1)}^{r} < 1$$
$$1 > x-1 > -1$$
$$2 > x > 0$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad , \quad x \in (0, 2)}$$

b) Hence find a power series for $\ln x$ centered at $x=1$

Solution

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad , \quad 0<x<2$$

$$\int_1^x \frac{1}{t}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{(t-1)^{n+1}}{n+1}\Bigg|_{t=1}^{t=x} \quad 0<x<2$$

$$\therefore \boxed{\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}} \quad , \quad 0<x<2$$

Theorem:

$$\int_a^x \sum_{n=0}^{\infty} c_n (t-a)^n\,dt = \boxed{\sum_{n=0}^{\infty} c_n \cdot \frac{(x-a)^{n+1}}{n+1}}$$

8) Given $\sum_{n=0}^{\infty} 5x^n = q$, solve for x in terms of q

Solution

$$\sum_{n=0}^{\infty} 5x^n = 5 + 5x + 5x^2 + \dots \quad , \quad |x|<1$$

$$q = \boxed{}$$

Fill in the blanks

$$\boxed{x = }$$

9) What conclusion can be reached by using the nth term test for divergence given the series

a) $\sum_{n=0}^{\infty} \frac{2n^2+13n+15}{n^3+8n^2+16n}$

Solution

The nth term of the series, a_n, is

$$a_n = \frac{2n^2+13n+15}{n^3+8n^2+16n}$$

The limit of a_n as $n \to \infty$ is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{2}{n} = 0$$

Therefore, the test is inconclusive.

nth term test for divergence:

If $a_n \to 0$ as $n \to \infty$, inconclusive

If $a_n \not\to 0$ as $n \to \infty$, $\sum_{n=0}^{\infty} a_n$ diverges

b) $\sum_{n=0}^{\infty} (-2)^n$

Solution

$$a_n = (-2)^n$$

$$\lim_{n\to\infty} a_n = (-2)^{\infty} \neq 0$$

(The limit does not exist)

Therefore, the infinite series $\sum_{n=0}^{\infty} (-2)^n$ diverges.

Answer to #8:

$$\sum_{n=0}^{\infty} 5x^n = 5 + 5x + 5x^2 + \dots, \quad |x|<1$$

$$q = \boxed{\frac{5}{1-x}}$$

$$\boxed{x = \frac{q-5}{q}}$$

Definition: Infinite Series

An **infinite series** is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots,$$

or

$$\sum_{k=1}^{\infty} a_k.$$

The numbers $a_1, a_2, \ldots$ are the **terms** of the series; a_n is the **nth term**.

The **partial sums** of the series form a sequence:

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$s_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

of real numbers, each defined as a finite sum.

If the sequence of partial sums has a limit S as $n \to \infty$, we say the series **converges** to the sum S, and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = S.$$

Otherwise, we say the series **diverges**.

The nth Term Test for Divergence

If the infinite series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_k + \cdots$$

converges, then

$$\lim_{k \to \infty} a_k = 0.$$

This means that if

$$\lim_{k \to \infty} a_k \neq 0,$$

the series must diverge.

The geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots \; = \; \sum_{n=1}^{\infty} ar^{\,n-1}$$

converges to the sum

$$\frac{a}{1-r} \quad \text{if } |r| < 1,$$

and diverges if $|r| \geq 1$.

$$-1 < r < 1$$

is the **interval of convergence**.

If $|x| < 1$, then the geometric series formula assures us that

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

Definition: Power Series

An expression of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

is a **power series centered at** $x = 0$.

An expression of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is a **power series centered at** $x = a$.

The term $c_n(x-a)^n$ is the n**th term**; the number a is the **center**.

Theorem 1: Term-by-Term Differentiation

If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

converges for $|x-a| < R$, then the series

$$\sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots,$$

obtained by differentiating the series for f term by term, converges for $|x-a| < R$ and represents $f'(x)$ on that interval. If the series for f converges for all x, then so does the series for f'.

Theorem 2: Term-by-Term Integration

If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

converges for $|x-a| < R$, then the series

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots + c_n\frac{(x-a)^{n+1}}{n+1} + \cdots,$$

obtained by integrating the series for f term by term, converges for $|x-a| < R$ and represents

$$\int_a^x f(t)\,dt$$

on that interval. If the series for f converges for all x, then so does the series for the integral.