8.2 L'Hospital's Rule

- Indeterminate forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty, \quad 1^{\infty}, \quad 0^{0}, \quad \infty^{0}$$

1) $$\lim_{\theta \to \frac{\pi}{2}} \frac{1-\sin\theta}{(\theta - \frac{\pi}{2})^2}$$

Solution

Direct substitution gives $\dfrac{1-\sin\frac{\pi}{2}}{(\frac{\pi}{2}-\frac{\pi}{2})^2} = \dfrac{0}{0}$

Since $\frac{0}{0}$, then we can apply $\overbrace{\text{L'Hospital's Rule}}^{\text{LHR}}$.

★ You must state the indeterminate form before using L'Hosp.

$$\lim_{\theta \to \frac{\pi}{2}} \frac{1-\sin\theta}{(\theta - \frac{\pi}{2})^2} = \lim_{\theta \to \frac{\pi}{2}} \frac{\text{Derivative of } (1-\sin\theta)}{\text{Derivative of } (\theta - \frac{\pi}{2})^2}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{-\cos\theta}{2(\theta - \frac{\pi}{2})}$$

★ L'Hospital's:

If $\dfrac{f(a)}{g(a)} = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

then $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Trying direct substitution again we get:

$$= \frac{-\cos\frac{\pi}{2}}{2(\frac{\pi}{2}-\frac{\pi}{2})} = \frac{0}{0}$$

So we apply LHR again

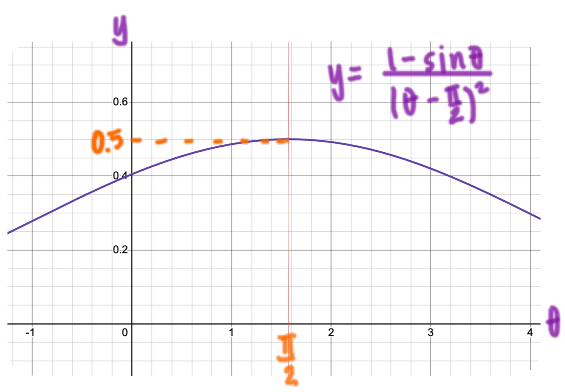
$$= \lim_{\theta \to \frac{\pi}{2}} \frac{\text{Deriv of } (-\cos\theta)}{\text{Deriv of } 2(\theta - \frac{\pi}{2})}$$

More formally:

$$\lim_{\theta \to \frac{\pi}{2}} \frac{\frac{d}{d\theta}(-\cos\theta)}{\frac{d}{d\theta}2(\theta-\frac{\pi}{2})}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{\sin\theta}{2}$$

$$= \frac{\sin\frac{\pi}{2}}{2} = \frac{1}{2}$$



2) $\lim_{x \to 0^-} \dfrac{\sin x}{x^2}$

Solution

$$\frac{\sin(0^-)}{(0^-)^2} = \frac{0^-}{0^+}$$

So we can use LHR:

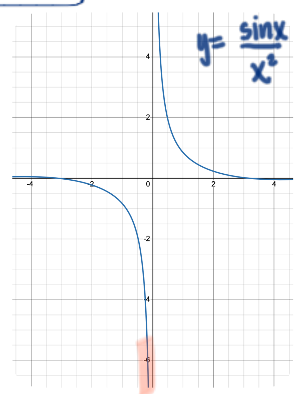
$$\lim_{x \to 0^-} \frac{\sin x}{x^2} = \lim_{x \to 0^-} \frac{\frac{d}{dx} \sin x}{\frac{d}{dx} x^2}$$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x}$$

$$= \frac{\cos(0^-)}{2(0^-)}$$

$$= \frac{1^-}{0^-}$$ Think: $\dfrac{0.999}{-0.001}$

$$= \boxed{-\infty}$$

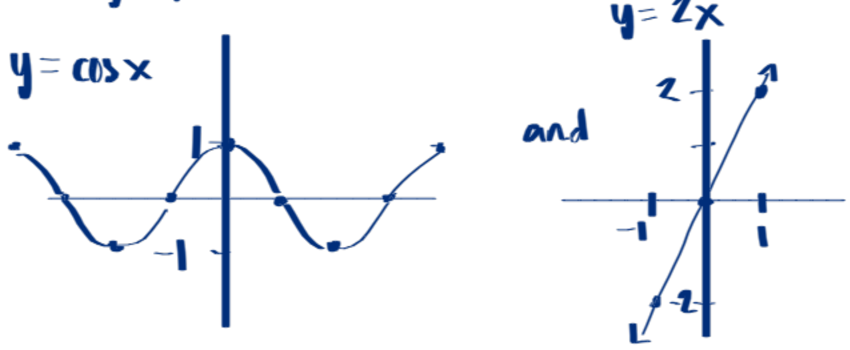


A common mistake with one-sided limits is to say $\frac{1}{0} = \infty$.

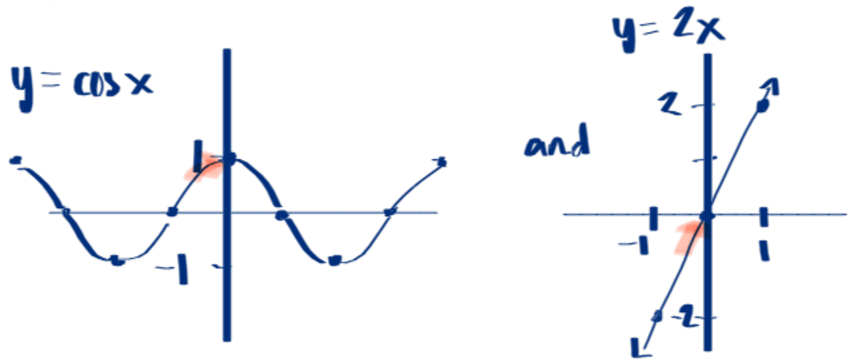
Check the sign of the numerator and denominator.

CAUTION! We are taking a one-sided limit here.

It helps greatly to know basic graphs:



As $x \to 0^-$ . . .



$\cos(0^-) \to 1$ from underneath, i.e. $0.9,\ 0.99,\ 0.999\ldots$

while $2x \to 0$ from underneath i.e. $-0.001,\ -0.00001, \ldots$

This is why I wrote $\dfrac{1^-}{0^-} = -\infty$

3) $\lim\limits_{x \to \infty} \dfrac{\ln(x+1)}{\log x}$

<u>Solution</u>

$$\frac{\ln(\infty+1)}{\log \infty} = \frac{\infty}{\infty}$$

So we use LHR

$$\lim_{x \to \infty} \frac{\ln(x+1)}{\log x}$$

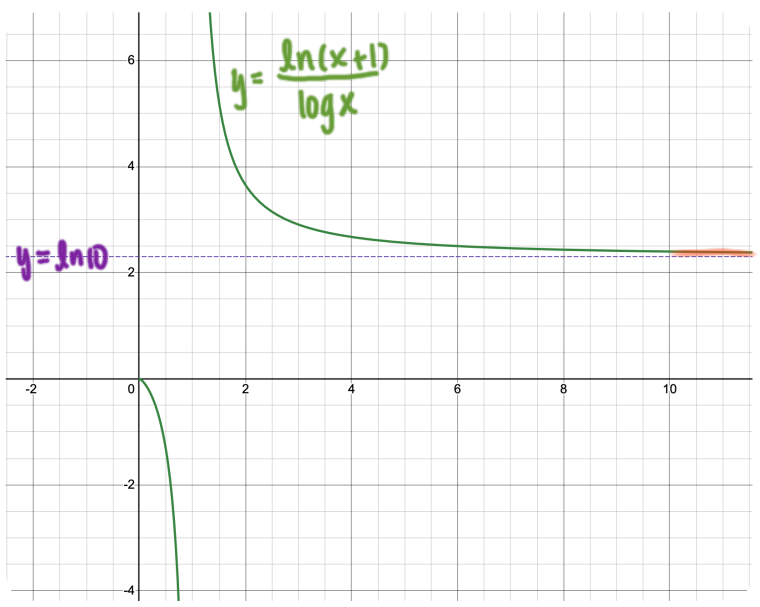
$$= \lim_{x \to \infty} \frac{\frac{1}{x+1}}{\frac{1}{x \ln 10}}$$

$$= \lim_{x \to \infty} \frac{x \ln 10}{x+1} = \frac{\infty}{\infty}$$

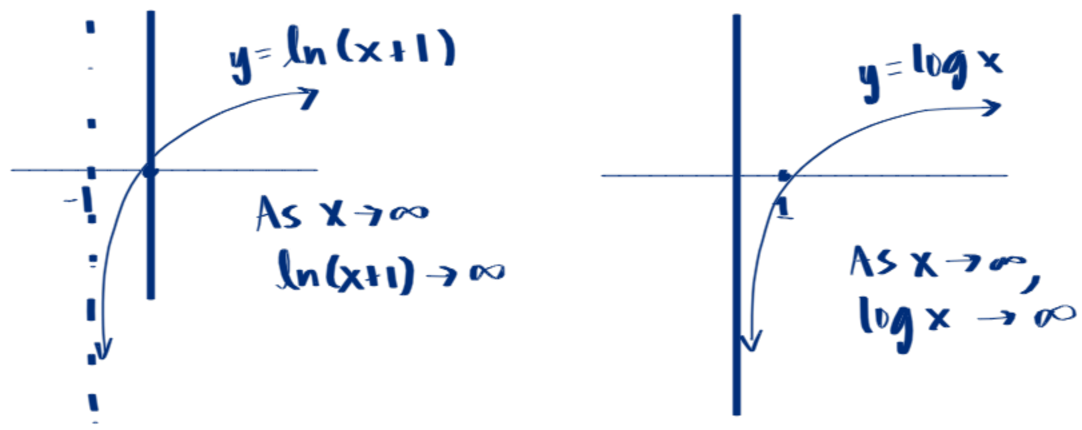
Use LHR again

$$= \lim_{x \to \infty} \frac{\ln 10}{1}$$

$$= \ln 10$$



Again, knowledge of basic graphs is so impt!



Alternatively,

$$\lim_{x \to \infty} \frac{x \ln 10}{x+1} = \ln 10 \cdot \lim_{x \to \infty} \frac{x}{x+1}$$

$$= \ln 10 \cdot \lim_{x \to \infty} \left(\frac{x}{x}\right) \to \text{End behavior model Ch. 1.2}$$

$$= \ln 10$$

4) $\lim_{x \to 0^+} x \ln x$

Solution

$0 \cdot \ln(0^+) = 0 \cdot (-\infty)$

use this technique

Rewrite $x \ln x = \frac{\ln x}{\frac{1}{x}}$

because $\frac{\ln(0^+)}{\frac{1}{0^+}} = \frac{-\infty}{\infty}$

So,

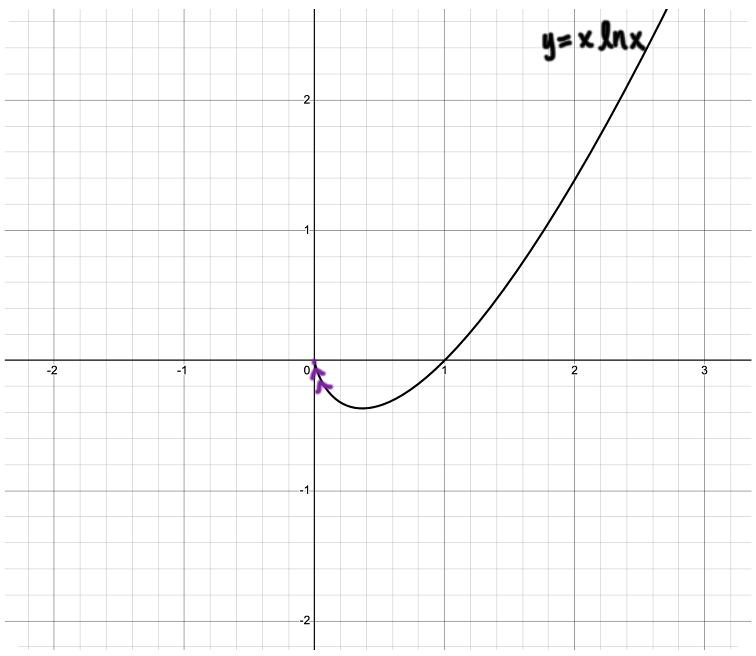
$$\lim_{x \to 0^+} x \ln x$$

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \qquad \text{(LHR)}$$

$$= \lim_{x \to 0^+} -x$$

$$= 0$$



5) $\lim_{x \to \infty} x \cdot \tan\left(\frac{1}{x}\right)$

Solution

$\infty \cdot \tan\left(\frac{1}{\infty}\right) = \infty \cdot \tan 0 = \infty \cdot 0$

Rewrite $x \cdot \tan\left(\frac{1}{x}\right) = \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$

because $\frac{\tan\left(\frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{0}{0}$ ✓

So,

$$\lim_{x \to \infty} x \cdot \tan\left(\frac{1}{x}\right)$$

$$= \lim_{x \to \infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\sec^2\left(\frac{1}{x}\right) \cdot -\frac{1}{x^2}}{-\frac{1}{x^2}} \qquad \text{(LHR)}$$

$$= \sec^2\left(\frac{1}{\infty}\right)$$

$$= \sec^2 0$$

$$= \left(\frac{1}{\cos 0}\right)^2 = 1$$

6) $\lim_{x \to 0^+} f(x)$, $\quad f(x) = \frac{1}{x} - \frac{1}{e^x - 1}$

Solution

$$\frac{1}{0^+} - \frac{1}{e^{0^+} - 1} = \infty - \frac{1}{1-1} = \infty - \infty$$

Combine $f(x)$ into a single fraction $\frac{e^x - 1 - x}{x(e^x - 1)}$

$$\frac{e^0 - 1 - 0}{0(e^0 - 1)} = \frac{0}{0}$$

If $\infty - \infty$ then combine into a single fraction

So,

$$\lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} \frac{e^x - 1 - x}{x(e^x - 1)}$$

LHR

$$= \lim_{x \to 0^+} \frac{e^x - 1}{xe^x + (e^x - 1)}$$

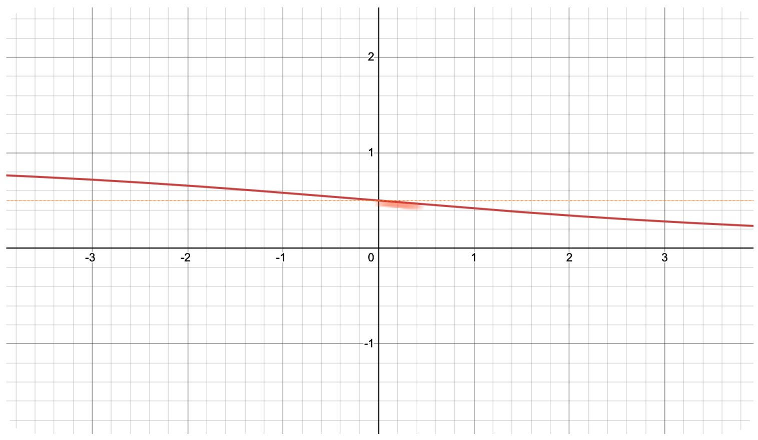
→ This is still $\frac{0}{0}$ via direct sub

LHR again

$$= \lim_{x \to 0^+} \frac{e^x}{xe^x + e^x + e^x}$$

$$= \frac{e^0}{0^+ + e^0 + e^0} = \frac{1}{2}$$

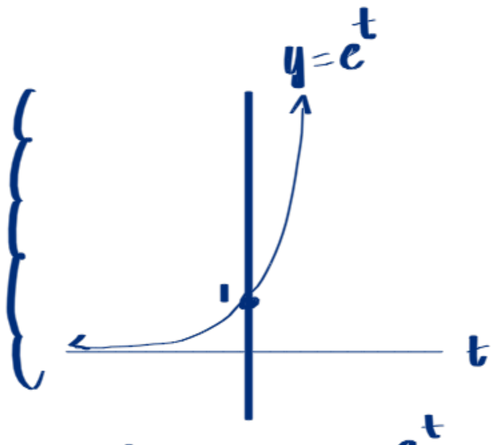
$y = \frac{1}{2}$



7) $\lim_{x\to\infty} f(x)$, $f(x) = x^{e^{-x}}$

Solution

$$\infty^{e^{-\infty}} = \infty^0$$



As $t \to -\infty$, $e^t \to 0$

$$\boxed{\lim_{x\to\infty} f(x) = e^{\lim_{x\to\infty} \ln f(x)}}$$

← know this and use it for the indet. forms 1^∞, ∞^0, 0^0

$$\ln f(x) = \ln x^{e^{-x}}$$

$$= e^{-x} \ln x$$

$$= \frac{\ln x}{e^x}$$

$$= \frac{\infty}{\infty} \text{ as } x \to \infty$$

so, $$\lim_{x\to\infty} f(x) = e^{\lim_{x\to\infty} \ln f(x)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln x}{e^x}}$$

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{e^x}} \quad \text{(LHR)}$$

$$= e^{\lim_{x\to\infty} \frac{1}{x \cdot e^x}}$$

$$= e^{\frac{1}{\infty}} = e^0 = 1$$

8) $\displaystyle\lim_{x \to 1} \frac{\int_1^x \frac{1}{t}\,dt}{x^3 - 1}$

Solution

$\displaystyle\frac{\int_1^1 \frac{1}{t}\,dt}{1^3 - 1} = \frac{0}{0}$

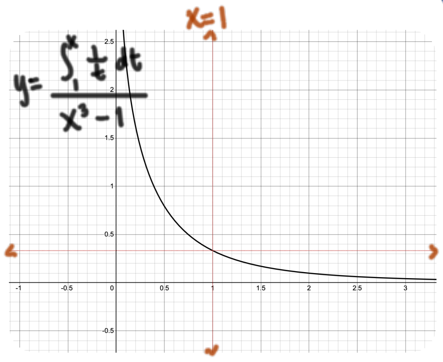
so, $\displaystyle\lim_{x \to 1} \frac{\int_1^x \frac{1}{t}\,dt}{x^3 - 1}$

$\displaystyle = \lim_{x \to 1} \frac{\frac{1}{x}}{3x^2}$ (LHR and the Fundamental Theorem of Calculus)

↳ Ch 5.4

$\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$\displaystyle = \lim_{x \to 1} \frac{1}{3x^3} = \frac{1}{3}$



$y = \frac{1}{3}$

9) $\displaystyle\lim_{x \to 0^+} \frac{x^2 + x}{-\ln x}$

Solution

$\displaystyle\frac{0^2 + 0}{-\ln 0} = \frac{0}{\infty} = 0$

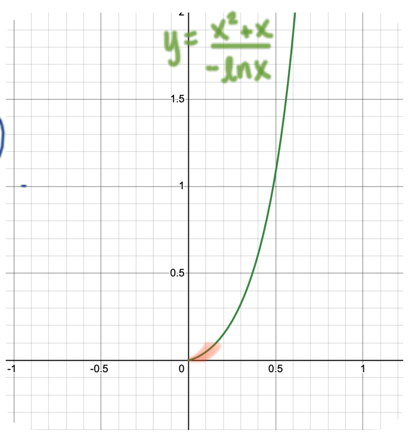
Yes, it's zero.

$\frac{0}{\infty}$ is NOT an indeterminate form

Why?

A small numerator and a large denominator produce the same effect $\longrightarrow$ shrink the fraction to 0.

$\displaystyle\therefore \lim_{x \to 0^+} \frac{x^2 + x}{-\ln x} = 0$



10) Let $f(x) = \begin{cases} x+2 & \text{, when } x \neq 0 \\ 0 & \text{, when } x = 0 \end{cases}$ and $g(x) = \begin{cases} x+1 & \text{, when } x \neq 0 \\ 0 & \text{, when } x = 0 \end{cases}$

Show that $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)} = 1$ but $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 2$

and explain why this does not contradict L'Hospital's.

Answer

$\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to 0} \dfrac{1}{1} = 1$. This part's easy

Now,

$\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{x+2}{x+1}$

$= 2$

This is because when taking limits, we don't use the function values at $x=0$.
Rather, we use the expressions for f & g near $x=0$

So we see a case where $\dfrac{f(0)}{g(0)} = \dfrac{0}{0}$ but $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} \neq \lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$.

Yet this does not violate LHR.

LHR states that f and g must be differentiable on an open interval I containing, in this case, 0. before using LHR.

f and g don't fit this requirement. (They are discontinuous at 0, thus, not differentiable on I.

So, LHR should not even be employed here.

L'Hospital's Rule

Suppose that

$$\frac{f(x)}{g(x)}$$

gives an indeterminate form of $\frac{0}{0}$ **or** $\frac{\infty}{\infty}$ as $x \to a$, and that f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ for $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the latter limit exists.

L'Hospital's Rule applies to one-sided limits as well.