8.1 Sequences

- Arithmetic and geometric sequences
- Writing recursive and explicit rules
- Finding the limit of the sequence
- Squeeze Theorem for sequences
- Absolute Value Theorem

1) Find the first 3 terms and the 8th term of the sequence:

$a_1 = 4$
$a_n = a_{n-1} - 2 \quad , \quad n \geq 2$

Solution

$a_1 = 4$

$a_2 = a_1 - 2$
$= 4 - 2$
$= 2$

$a_3 = a_2 - 2$
$= 2 - 2$
$= 0$

$\boxed{\{a_n\} = \{4, 2, 0, -2, \ldots\}}$

$a_n = \underbrace{a_{n-1} - 2}$

"each term is 2 less than the previous term"

So it is an arithmetic sequence

↳ a sequence where each term is equal to the previous term + d

Ex: $10 \xrightarrow{+(-3)} 7 \xrightarrow{+(-3)} 4 \xrightarrow{+(-3)} 1 \rightarrow d = -3$

Since the sequence is arithmetic,

$\boxed{a_8 = a_1 + 7d}$ → Add 7 d's to the first term to arrive at the 8th term

$a_8 = 4 + 7(-2)$
$= 4 - 14$
$= -10$

We can go straight from a_1 to a_8.

2) $u_1 = -5$, $u_2 = 2$

$u_n = u_{n-1} + u_{n-2}$, $n \geq 3$

Find the first 3 terms and the 8th term.

Solution

u_n → "each term

$=$ → is

$u_{n-1} + u_{n-2}$ → the sum of the 2 previous terms"

$u_3 = u_2 + u_1$
$= 2 + -5$
$= -3$

$\boxed{\{u_n\} = \{-5,\ 2,\ -3,\ -1,\ \ldots\}}$

Since this sequence is neither arithmetic nor geometric, we need to list all terms until we arrive at u_8.

$-5,\ 2,\ -3,\ -1,\ -4,\ -5,\ -9,\ -14$

$\underset{u_8}{\parallel}$

3) Given 10, 27, 44, 61, ...

Find a recursive rule and an explicit rule for the nth term

Solution

The sequence is arithmetic with common difference of 17.

$$27-10 = 44-27 = 61-44$$

$$d = 17$$

So,

recursive statement

"each term is the prev. term + 17"

$$\boxed{\begin{cases} a_n = a_{n-1} + 17 \\ a_1 = 10 \end{cases}}$$

Don't forget to initialize/give a starting value.

explicit statement

Add (n-1) 17's to the first term to arrive at the nth term

$$\boxed{a_n = a_1 + (n-1)17}$$

$$a_n = 10 + (n-1)17$$

$$a_n = 10 + 17n - 7$$

$$\boxed{a_n = 17n + 3}$$ (simplified)

4) Given 1.5, 2.25, 3.375...

Find a recursive and explicit rule for the sequence.

Solution

$1\frac{1}{2}$, $2\frac{1}{4}$, $3\frac{3}{8}$

$\frac{3}{2}$, $\frac{9}{4}$, $\frac{27}{8}$

This is a geometric sequence with common ratio 1.5

$$\frac{\frac{9}{4}}{\frac{3}{2}} = \frac{\frac{27}{8}}{\frac{9}{4}} = \frac{3}{2}$$

Therefore

recursive

"each term is the prev. term × 1.5"

$$\boxed{\begin{cases} a_n = a_{n-1} \cdot 1.5 \\ a_1 = 1.5 \end{cases}}$$

explicit

You can multiply 1.5 (n−1) times to the first term to arrive at the nth term

$$a_n = a_1 \cdot \overbrace{(1.5)(1.5)\cdots(1.5)}^{n-1 \text{ factors}}$$

$$\boxed{a_n = a_1 (1.5)^{n-1}}$$

$$a_n = 1.5\,(1.5)^{n-1}$$

$\boxed{a_n = 1.5^n}$ (simplified)

5) Let $\{a_n\}$ be an arithmetic sequence where $a_5 = 5$ and $a_9 = -3$. Find an explicit rule for a_n.

<u>Solution</u>

Goal: $\boxed{a_n = a_1 + (n-1)d}$

we need to find a_1 & d

<u>Method 1: Systems</u>

$$\begin{cases} 5 = a_1 + 4d \\ -3 = a_1 + 8d \end{cases}$$

$$\begin{array}{rcl} -3 &=& a_1 + 8d \\ -5 &=& -a - 4d \\ \hline -8 &=& 4d \end{array}$$

$-2 = d$

$5 = a_1 + (-8)$

$a_1 = 13$

Method 2:

We added 4 d's to 5 to arrive at -3

$$5 \overset{+d}{,} \_ \overset{+d}{,} \_ \overset{+d}{,} \_ \overset{+d}{,} -3$$

$5 + 4d = -3$

$4d = -8$

$d = -2$

We added 4 (-2)'s to a_1 to arrive at a_5

$a_1 + 4(-2) = 5$

$a = 13$

Therefore,

$\boxed{a_n = 13 + (n-1)(-2)}$

or $\boxed{a_n = -2n + 15}$ (simplified)

6) Given $\{u_n\}$ is a geometric sequence with $a_6 = 3010$ and $a_9 = 3{,}010{,}000$. Find an explicit rule.

Solution

Goal: $\boxed{a_n = a_1 \cdot r^{n-1}}$

To find a_1 and r...

Method of choice: Systems of equations

$$\begin{cases} 3010 = a_1 \cdot r^5 \\ 3010000 = a_1 \cdot r^8 \end{cases}$$

Other methods for solving for a_1 & r are welcome as long as the reasoning is valid. ☺

$$\frac{3010000}{3010} = \frac{a_1 \cdot r^8}{a_1 \cdot r^5}$$

$$1000 = r^3$$

$$\sqrt[3]{1000} = r$$

$$r = 10$$

$$3010 = a_1 \cdot 10^5$$

$$a_1 = \frac{3010}{10^5}$$

$$a_1 = 0.0301$$

$$\therefore \boxed{a_n = 0.0301 \cdot 10^{n-1}}$$

7) Find the limit of $a_n = \frac{2n+1}{n}$

Solution

$$\lim_{n\to\infty} \frac{2n+1}{n}$$ $\left(\frac{\infty}{\infty}\right.$ via direct subt$\left.\right)$

$$= \lim_{n\to\infty} \frac{2\cancel{n}}{\cancel{n}} + \frac{1}{n}$$

$$= 2 + \frac{1}{\infty}$$

$$= 2 + 0 = 2$$

That is, as the sequence progresses, the terms converge to 2.

these terms are ≈ 2.

$$\left\{ 3, 2\tfrac{1}{2}, 2\tfrac{1}{3}, 2\tfrac{1}{4}, 2\tfrac{1}{5}, \ldots, 2\tfrac{1}{1000000}, \ldots, \ldots \right\}$$

8) Find the limit of $a_n = (-1)^n \frac{n+1}{n^2+2}$

Solution

$$\lim_{n\to\infty} (-1)^n \cdot \frac{n+1}{n^2+2}$$

(We cannot take lim of each factor separately since $\lim_{n\to\infty} (-1)^n$ does not exist)

$$(-1)^n = \{-1, 1, -1, 1, -1, \ldots\}$$

We need to use
the Squeeze Theorem for Sequences

First, $\quad -1 \le (-1)^n \le 1$

$$\boxed{-\frac{n+1}{n^2+2} \le (-1)^n \frac{n+1}{n^2+2} \le \frac{n+1}{n^2+2}}$$

$\left(\text{Note: } \frac{n+1}{n^2+2} > 0\right)$

Next,

(Recall: End behavior model in Ch. 1.2)

$$\lim_{n\to\infty} -\frac{n+1}{n^2+1} = \lim_{n\to\infty} -\frac{n}{n^2} = -\frac{1}{\infty} = 0$$

same value.

and
$$\lim_{n\to\infty} \frac{n+1}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2} = \frac{1}{\infty} = 0$$

Therefore, by the Squeeze Theorem,

$\lim_{n\to\infty} (-1)^n \cdot \frac{n+1}{n^2+2}$ is also 0.

9) Find the limit of $a_n = \cos\left(n\frac{\pi}{2}\right)$

<u>Solution</u>

We write out cosine values of multiples of $\frac{\pi}{2}$

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	
$\{0,$	$-1,$	$0,$	$1,$	$0,$	$-1,$	$0,$	$\ldots\}$

As we can see, the sequence is divergent.

(i.e. it doesn't settle at a single value)

10) Use the Absolute Value Theorem to answer # 8.

Solution

First, $\left|(-1)^n \frac{n+1}{n^2+2}\right| = \frac{n+1}{n^2+2}$

If $\lim_{n\to\infty} |a_n| = 0$

then $\lim_{n\to\infty} a_n = 0$

Next, $\lim_{n\to\infty} \frac{n+1}{n^2+2} = \lim_{n\to\infty} \frac{n}{n^2} = \frac{1}{\infty} = 0$

Therefore, by the Abs Val Th,

$\lim_{n\to\infty} (-1)^n \cdot \frac{n+1}{n^2+2}$ is also 0

A sequence $\{a_n\}$ is an **arithmetic sequence** if it can be written in the form

$$\{a,\, a+d,\, a+2d,\, \ldots,\, a+(n-1)d,\, \ldots\}$$

for some constant d. The number d is the **common difference**.

Each term in an arithmetic sequence can be obtained recursively from its preceding term by adding d:

$$a_n = a_{n-1} + d \quad \text{for all } n \geq 2.$$

A sequence $\{a_n\}$ is a **geometric sequence** if it can be written in the form

$$\{a,\, ar,\, ar^2,\, \ldots,\, ar^{n-1},\, \ldots\}$$

for some nonzero constant r. The number r is the **common ratio**.

Each term in a geometric sequence can be obtained recursively from its preceding term by multiplying by r:

$$a_n = a_{n-1} \cdot r \quad \text{for all } n \geq 2.$$

Limit of a Sequence

A sequence $\{a_n\}$ has a limit L as $n \to \infty$ if the terms of the sequence get closer and closer to L as n becomes very large.

In other words, as you go further along the sequence, the values of a_n approach a fixed number L.

We write:

$$\lim_{n\to\infty} a_n = L$$

and say that the sequence **converges** to L.

If the sequence does not approach any single value, we say it **diverges**.

The Squeeze Theorem for Sequences

If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

and there exists an integer N such that

$$a_n \leq b_n \leq c_n \quad \text{for all } n > N,$$

then

$$\lim_{n\to\infty} b_n = L.$$

Absolute Value Theorem

Consider the sequence $\{a_n\}$. If

$$\lim_{n\to\infty} |a_n| = 0,$$

then

$$\lim_{n\to\infty} a_n = 0.$$